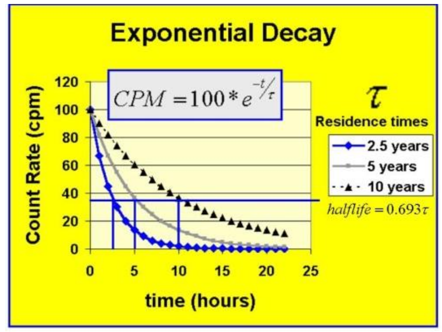
Unit 5



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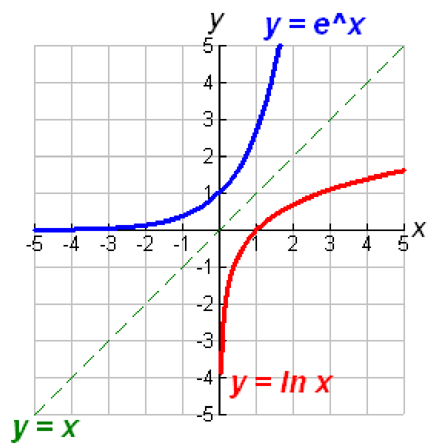
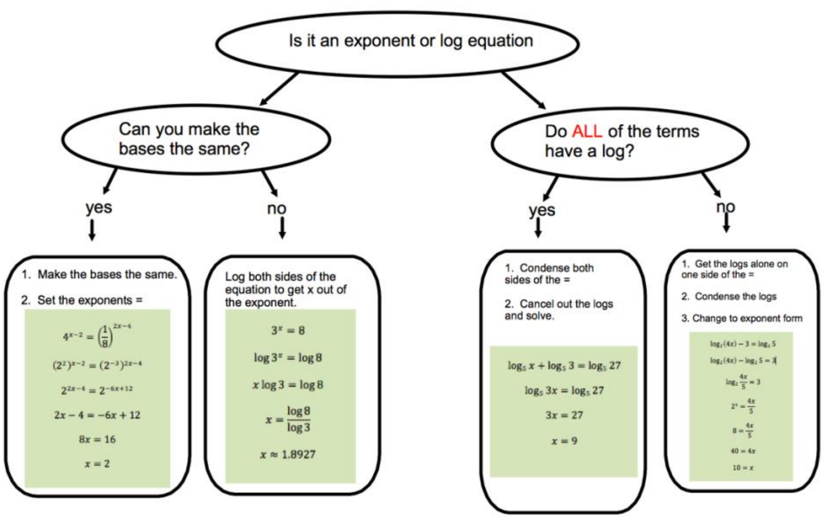




5.1 Inverses

The word inverse is related to the word invert meaning to reverse, turn upside down, to do the opposite. In mathematics, an inverse function is a function that undoes a function. Remember that a function is a rule that assigns exactly one output value to each input value. The inverse is a rule that will take the output value and give back the original input value. All relations (not just functions) have an inverse (which may or may not be a function). If the inverse relation is a function, the notation is $f^{-1}(x)$.

For example, suppose you are interviewing for a job at a telemarketing firm that pays $10 per hour for as many hours as you wish to work, and the firm pays you at the end of each day. If you are trying to calculate your potential salary, the rule is **Salary = $10 times the number of hours you worked in a day**. If you were to graph the rule, all the points on the graph would consists of pairs of numbers such as (number of hours worked, salary for hours worked). Since you can't work less than zero hours or more than 24 hours in a day, the only numbers in the domain are the numbers between and including 0 to 24. How about the range? You cannot earn less than 0 dollars and you cannot earn more than $240 dollars.

The inverse function undoes this function. Suppose you were told that Sal earned $140 dollars on Tuesday, could you work backwards and find the number of hours he worked? The inverse rule would be **Number of Hours Worked = Salary / 10**. If you divide Sal's salary of $140 by 10, you would know that he worked 14 hours on Tuesday. The first rule of Salary = $10 x Number of Hours worked is a function. The undo rule of Number of Hours Worked = Salary / 10 is the inverse function. Notice that points on the graph of the inverse function would consist of pairs of numbers such as (salary for hours worked, number of hours worked). The input (domain) values of the original function have become the output (range) values of the inverse function and vice versa.

If the relation is given in table form, then the inverse of the relation is found by interchanging the input and output columns of the table.

Examples:

1. The table gives the height of a person at different ages. As each input (age) has a unique output (height), the table is a function. Find the inverse of the function.

Age (years)	Height (inches)
5	40
9	45
13	60
17	66
19	66

The inverse will give the age of a person for a particular height. The input (domain) of the inverse relation will be height and the output (range) will be age.

Height (inches)	Age (years)
40	5
45	9
60	13
66	17
66	19

Notice that the inverse relation is not a function because for a height (input) of 66 inches there are two output values of 17 and 19 years old.

2. The following table gives your monthly payment of a car loan if you borrow $25000 at 4% interest for m months. Find the inverse.

m, months to repay loan	12	24	36	48	60
P, monthly payment	2128.75	1085.62	738.10	564.48	460.41

The inverse will find the months needed to repay the loan (output) given the monthly payment (input).

P, monthly payment	2128.75	1085.62	738.10	564.48	460.41
m, months to repay loan	12	24	36	48	60

Notice that both the original relation and the inverse relation are functions since each input has exactly one unique output.

If the relation is given in words, then to find the inverse we need to undo each operation in reverse order.

Examples:

1. If the function rule is to take a number, multiply it by 4 and then add 7 to get the output, find the inverse in words.

   The inverse must undo each of these steps in reverse order. The last step was to add 7 so to undo this we must subtract 7. To undo multiplying by 4, we would divide by 4. So the inverse is to take a number, subtract 7 and then divide by 4. Let's check to see if this works.

   Picking any number and applying the function rule:

   Pick a number: 5 (input)
   Multiply by 4: 20
   Add 7: 27 (output)

   The inverse should start with the output of the function and get us back to the original input.

   Starting with the output: 27 (input)
   Subtracting 7: 20
   Dividing by 4: 5 (output of the inverse and input of the original function)

2. If the function rule is to take an input number, subtract 6, then take the cube root to find the output, find the inverse in words.

   The inverse must first undo the cube root which can be done by cubing the value. To undo subtraction, we add. Therefore, the inverse rule is to take a number, cube the number, then add 6 to get the output.

Check:

Pick a number (input): 14
Subtract 6: 8
Take cube root: 2

Inverse starts with an input of 2
Cube the number: 8
Add 6: 14 which is the original number chosen.

If the relation is given as a formula, the inverse can be found in the same way as if it was given in words or we can follow the steps given below.

Steps to find an inverse from a formula:

1. If the function is written using f(x), then replace f(x) with y.
2. Interchange the variables (x becomes y and y becomes x).
3. Solve for y.
4. If the result is a function, then replace y with $f^{-1}(x)$.

Examples:

1. Given f(x) = 6x + 1, find the inverse.

   Following the steps above:
   Rewrite using y instead of f(x): y = 6x + 1
   Interchange the variables: x = 6y + 1
   Solve for y: x – 1 = 6y
   $\frac{x-1}{6} = y$
   Rewrite in function notation: $f^{-1}(x) = \frac{x-1}{6}$

2. Given $f(x) = 2x^5 - 11$, find the inverse.

   Following the steps above:
   Rewrite using y instead of f(x): $y = 2x^5 - 11$
   Interchange the variables: $x = 2y^5 - 11$
   Solve for y: $x + 11 = 2y^5$
   $\frac{x+11}{2} = y^5$
   $\sqrt[5]{\frac{x+11}{2}} = y$
   Rewrite in function notation: $f^{-1}(x) = \sqrt[5]{\frac{x+11}{2}}$

3. Given f(x) = $\frac{x^2+2}{7}$, find the inverse.

Following the steps above:

Rewrite using y instead of f(x): y = $\frac{x^2+2}{7}$

Interchange the variables: x = $\frac{y^2+2}{7}$

Solve for y: 7x = y^2 + 2

7x – 2 = y^2

$\pm\sqrt{7x-2}$ = y

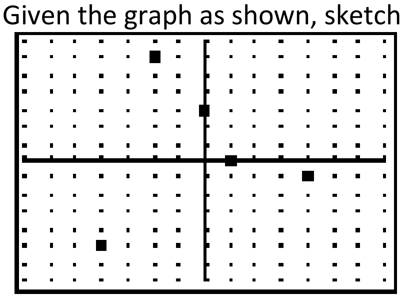
As a single input value can result in two output values, the inverse is <u>not</u> a function so we cannot use the notation for an inverse function.

Note that in some cases we might want to restrict the domain of a function so that its inverse is a function. Also, in certain applications, the domain has restrictions which we need to take into account when finding an inverse. For example, given the function f(x) = x^2, the inverse would be y = $\pm\sqrt{x}$ which is not a function; but, if we restrict the initial domain of f(x) to values greater than or equal to zero, then the range of the inverse must be greater than or equal to zero and thus, the inverse would be y = $\sqrt{x}$ which is a function.

If the relation is given as a graph, the inverse of the relation can be found by interchanging the input and output values of each point to graph the inverse. The graph of a relation and its inverse are symmetric about the line y = x. This means that is you folded the graph along the line y = x, the graphs would fold onto each other.

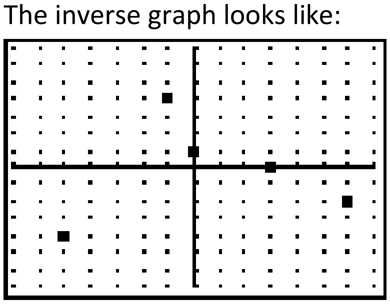
Examples:

1. Given the graph as shown, sketch a graph of its inverse. Assume the scale on both axes is 1 unit.



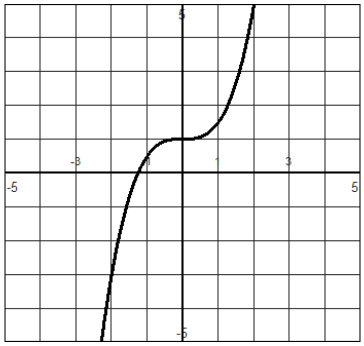
This graph has the points (-4, -5), (-2, 6), (0, 3), (1, 0) and (4, -1).

The inverse graph looks like:

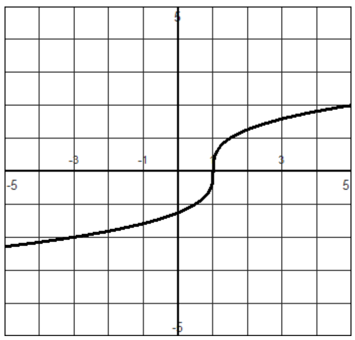


The inverse graph has the points (-5, -4), (6, -2), (3, 0), (0, 1), and (-1, 4).

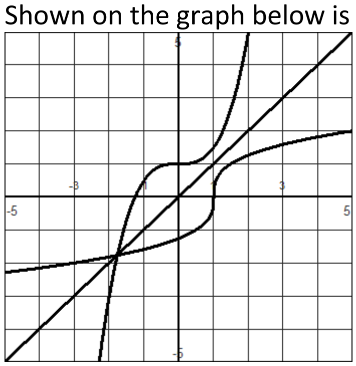
2. Given the graph as shown, sketch a graph of its inverse. Assume the scale on both axes is 1 unit.



The inverse can be found by reversing several key points on the graph and then sketching the inverse. It can also be found by reflecting the graph over the line y = x.



Shown on the graph below is the original graph, the line y = x, and the graph of the inverse.

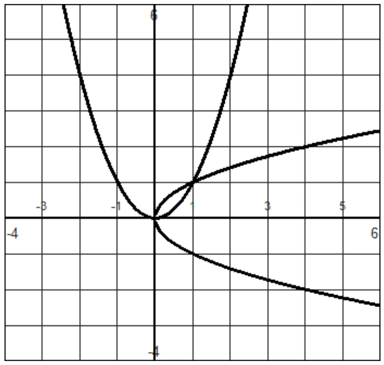


If a function has an inverse that is also a function, the function is said to be **one-to-one**. To decide whether the graph of a function has an inverse which is a function we can use the horizontal line test.

Horizontal line test:
If no horizontal line intersects the graph of a function more than once, then the inverse is also a function.

The graph in the previous example passes the horizontal line test so its inverse is also a function (which we can see from its graph) and the function is one-to-one.

The graph of $f(x) = x^2$ would not pass the horizontal line test so its inverse will not be a function and $f(x) = x^2$ is not one-to-one. The graph on the next page shows the graph of $f(x) = x^2$ and its inverse relation.



Definition:
If f is a one-to-one function, then f^{-1} is the inverse function of f if:

1. $(f \circ f^{-1})(x) = f(f^{-1}(x)) = x$ for every x in the domain of f^{-1} and
2. $(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x$ for every x in the domain of f.

Example: Verify that the inverse of $f(x) = 2x^3 - 1$ is $f^{-1}(x) = \sqrt[3]{\frac{x+1}{2}}$ by using the definition of one-to-one.

We need to check $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$.

$$(f \circ f^{-1})(x) = 2\left(\sqrt[3]{\frac{x+1}{2}}\right)^3 - 1 = 2\left(\frac{x+1}{2}\right) - 1 = x$$

$$(f^{-1} \circ f)(x) = \sqrt[3]{\frac{(2x^3 - 1)+1}{2}} = x$$

Homework:

Find the inverse relations of the following functions. Is the inverse a function?

1.

x	F(x)
-1	2
0	5
1	-3
2	7
3	1

2.

x	F(x)
2	0
4	-1
7	-2
9	0
12	-1

3.

Time (seconds)	Height (feet)
1	3
2	51
3	67
4	51

4. Given the table as shown, find the table for the inverse and use the information to find $g^{-1}(5)$.

t	g(t)
-1	11
3	-7
5	1
8	5
10	-1

5. When a layer of ice forms on a pond, the thickness of the ice, d (in centimeters), varies directly as the square root of time (in minutes). The table shows the ice thickness at certain times. Make a table for the inverse.

t (min)	10	30	40	60
d (cm)	0.5	0.87	1.01	1.24

6. A. Given f(-3) = 5, f(-2) = 0, f(0) = 4, and f(2) = 7, find the inverse function values.

   B. What is $f^{-1}(0)$?

   C. What is $f^{-1}(7)$?

Find the inverse of each of the following equations.

7. f(x) = 4x + 9

8. $g(x) = x^7 + 5$

9. $f(x) = 3x^3 - 11$

10. $h(t) = \sqrt{3t + 5}$

11. $g(m) = \frac{m-3}{8}$

12. $f(d) = 3\sqrt[4]{x} + 1$

13. $h(x) = 6x^2 - 5$

14. $Y = \sqrt[3]{\frac{2x-3}{4}}$

15. The function $C(F) = \frac{5}{9}(F - 32)$ gives the Celsius temperature in terms of the Fahrenheit temperature.

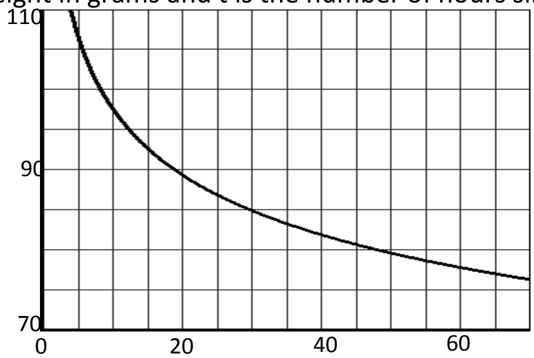
   A. Find C(75) and explain its meaning in context.

   B. Find the Fahrenheit temperature if the Celsius temperature is 28°C.

   C. Find the formula for the inverse.

   D. Write the answer to part B using inverse function notation.

16. The graph shows the weight of a bat which drops between meals. The graph is W(t) where W is the weight in grams and t is the number of hours since its last meal.



A. Estimate W(40) and explain what it means in context.

B. Estimate $W^{-1}(80)$ and explain what it means in context.

17. The speed s (in miles per hour) a car was traveling if it skidded d feet after the brakes were applied on a dry concrete road is given by $s = \sqrt{24d}$. Approximate the speed to the nearest mile per hour.

A. Fill in the table below.

Skid distance, d	Speed, s
20	
35	
80	

B. Make a table of that shows the inverse of the table in part A.

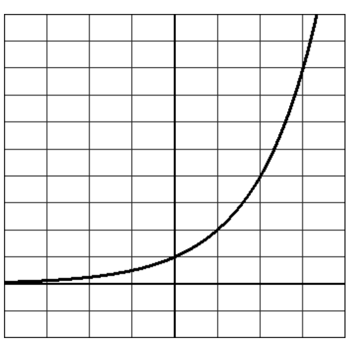
C. Write a formula for the inverse of s.

D. Find $s^{-1}(70)$ and explain what it means in context.

E. Find s(70) and explain what it means in context.

Given the graphs shown, sketch a graph of the inverse. Is the inverse a function?

18.



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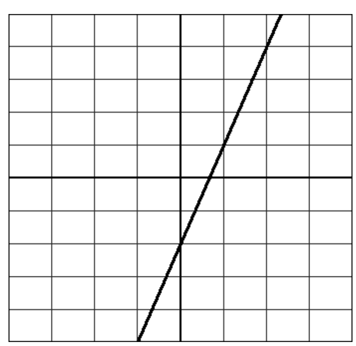

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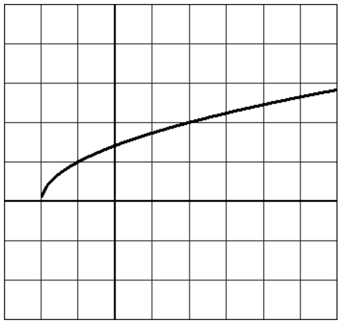

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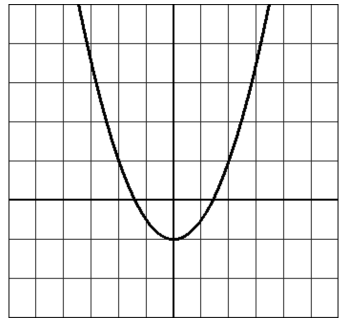

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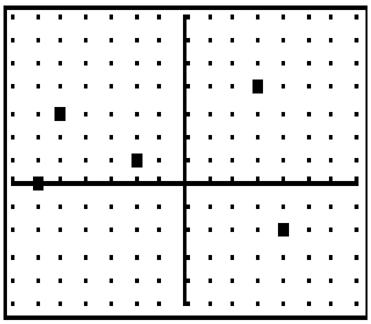

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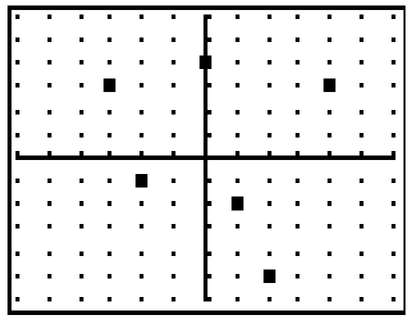

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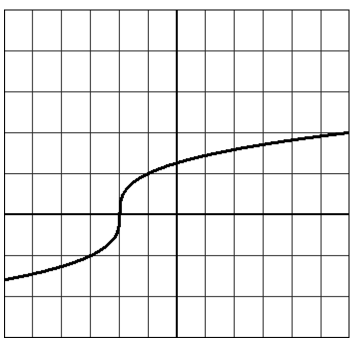

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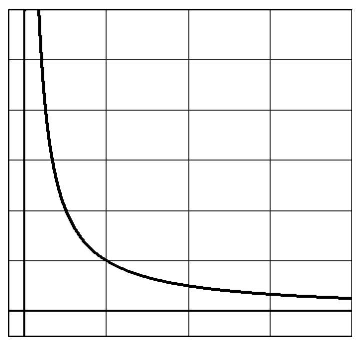


26. Which of the graphs in 18-25 represent one-to-one functions?

27. Verify that $f^{-1}(x) = 3x + 1$ is the inverse of $f(x) = \frac{1}{3}x - \frac{1}{3}$ by showing that $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$.
28. Verify that the inverse function in number 12 is the inverse of the given function by showing that $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$.
29. Verify that the inverse function in number 14 is the inverse of the given function by showing that $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$.
30. Restrict the domain of $f(x) = 6 - x^2$ so that f is one-to-one.
31. Restrict the domain of $f(x) = 3(x + 1)^2$ so that f is one-to-one.

5.2 Exponential Growth and Decay

Many quantities grow (or decrease) at a rate that is proportional to the current value of the quantity. The number of micro-organisms in a culture will increase exponentially until an essential nutrient is exhausted. The population of the United States of America is exponentially increasing at an average rate of one and a half percent a year (1.5%). In economics, the U.S. GDP per capita has grown at an exponential rate of approximately two percent per year for two centuries. Multi-level marketing, carbon dating, compound interest, radioactive decay, and computer processing power are just a few other applications that involve exponential growth or decay.

Exponential growth can be defined as the growth of a system in which the amount being added to the system is proportional to the amount already present: the bigger the system is, the greater the increase.

The function $P(t) = P_0b^t$ models exponential growth where P_0 is the initial value of P and b is called the growth factor ($b > 1$).

Let's look at a couple of examples of exponential growth.

Examples:

1. In a laboratory experiment, there are 50 bacteria in a petri dish. The number of bacteria doubles every hour.
   A. Fill in the table showing the population P(t) of bacteria after t hours.

t	0	1	2	3	4	5
P(t)						

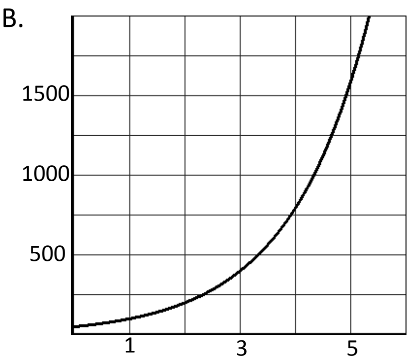
   B. Plot the points and connect them in a smooth curve.
   C. Write a function for P(t) that gives the number of bacteria at any time t.
   D. Use the function to find out how many bacteria there are after 3.5 hours. How many are there after 1 day?

Solution:

A. Fill in the table showing the population P(t) of bacteria after t hours.

t	0	1	2	3	4	5
		$50 \cdot 2^1$	$50 \cdot 2^2$	$50 \cdot 2^3$	$50 \cdot 2^4$	$50 \cdot 2^5$
P(t)	50	100	200	400	800	1600

B.



C. The initial value of the population is 50 and the population is doubling (being multiplied by 2) every hour. The formula will be $P(t) = 50(2)^t$.

D. After 3.5 hours there will be $P(3.5) = 50(2)^{3.5} = 566$ bacteria (rounded to the nearest whole number). After 1 day, which is 24 hours, there will be $P(24) = 50(2)^{24} = 838,860,800$ bacteria.

2. A pet store starts with 5 mice. The population of mice quadruples every 3 months.
   A. Fill in the table showing the population P(t) of mice after t months.

t	0	3	6	9	12
P(t)					

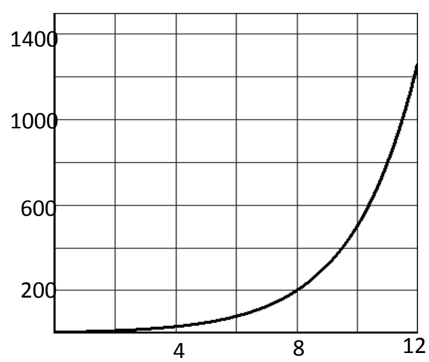
   B. Plot the points and connect them in a smooth curve.
   C. Write a function for P(t) that gives the number of mice at any time t.
   D. Use the function to find out how many mice there are after 7 months.
   E. Make a graph of the population of mice on your graphing calculator. Use this graph to estimate how many months it will take to have 500 mice.

Solution:

A.

t	0	3	6	9	12
		$5 \cdot 4^1$	$5 \cdot 4^2$	$5 \cdot 4^3$	$5 \cdot 4^4$
P(t)	5	20	80	320	1280

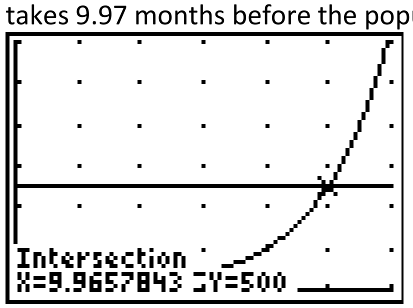
B.



C. The initial population is 5 mice. It takes 3 months for the population to triple so the formula is $P(t) = 5(4)^{t/3}$.

D. $P(7) = 5(4)^{7/3} = 127$ mice.

E. Graphing $y_1 = 5(4)^{x/3}$ and $y_2 = 500$. Then, using the intersection command, we find that it takes 9.97 months before the population reaches 500 mice.



3. Interest on a savings account is compounded annually. Assume we invest $1000 at an interest rate of 4%.
   A. Write a function for A(t) that gives the amount of money in the account after t years.
   B. Use the function to find out how much money is in the account after 10 years.

Solution:

A. The initial amount in the account is $1000. The interest rate is 4% but the growth rate is 100% + 4% = 104% because we have the original amount in the account plus the interest amount. The formula is $A(t) = 1000(1.04)^t$.

B. $A(10) = 1000(1.04)^{10}$ = $1480.24

Compound interest

The amount A(t) of money in an account earning interest at an annual interest rate of r is

$A(t) = P(1 + \frac{r}{n})^{nt}$

where P is the principal, r is the interest rate as a decimal, n is the number of times interest is compounded in a year, and t is time in years.

Example:

If $700 is invested in an account which accrues 3% interest compounded monthly, how much is in the account after 5 years?

Solution:

If interest is compounded monthly, interest is accrued 12 times a year (n = 12).

$A(t) = 700(1 + \frac{.03}{12})^{12t}$

After 5 years, $A(5) = 700(1 + \frac{.03}{12})^{12(5)}$ = $813.13.

A quantity is said to be subject to **exponential decay** if it decreases at a rate proportional to its value. Exponential decay decreases most rapidly at first and then more slowly as the amount decreases. Examples of exponential decay include radioactive decay, carbon dating, and depreciation.

The function $P(t) = P_0b^t$ models exponential decay where P_0 is the initial value of P and b is called the growth (decay) factor ($0 < b < 1$).

Let's look at some examples of exponential decay.

Examples:

1. Each year the local country club sponsors a tennis tournament. Play starts with 128 participants. During each round, half of the players are eliminated.

   A. Fill in the table showing the number P(t) of players after t rounds.

t	0	1	2	3	4
P(t)					

   B. Plot the points and connect them in a smooth curve.

   C. Write a function for P(t) that gives the number of players after t rounds.

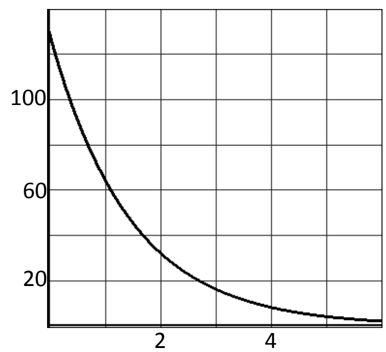
   D. Use the function to find how many players remain after 5 rounds.

Solution:

A.

t	0	1	2	3	4
		128(1/2)	$128(1/2)^2$	$128(1/2)^3$	$128(1/2)^4$
P(t)	128	64	32	16	8

B.



C. The initial amount is 128 players and this number decreases by ½ every round. The formula for the number of players is $P(t) = 128(\frac{1}{2})^t$.

D. After 5 rounds, $P(5) = 128(\frac{1}{2})^5 = 4$ players.

2. Atmospheric pressure as a function of altitude is an exponential decay function. For every 1000 meters increase of elevation, the atmospheric pressure is reduced by 11.5%. Assume that the *sea-level pressure* is 101.325 kPa.
   A. Fill in the table showing the pressure at different altitudes in meters.

a, meters	0	1000	2000	3000	4000
P(a)					

   B. Plot the points and connect them in a smooth curve.
   C. Write a function for P(a) that gives the atmospheric pressure at m meters.
   D. Use the function to find the atmospheric pressure at an altitude of 1700 meters.

Solution:

A. The atmospheric pressure decreases by 11.5% for every 1000 meter increase in altitude. We are interested in the amount of pressure remaining (not the amount of decrease) which will be 100%-11.5% = 88.5%. So, for each 1000 meter increase in altitude, there is 88.5% of the pressure remaining.

a, meters	0	1000	2000	3000	4000
		101.325(.885)	$101.325(.885)^2$	$101.325(.885)^3$	$101.325(.885)^4$
P(a)	101.325	89.673	79.36	70.234	62.157

B.

C. $P(a) = 101.325\ (0.885)^{a/1000}$ because the initial pressure is 101.325kPa. The base is 0.885 since if the pressure decreases by 11.5% then there is 88.5% remaining. The exponent is divided by 1000 since the pressure decreases by the given amount for every 1000 meter change in altitude.

D. $P(1700) = 101.325\ (0.885)^{1700/1000} = 82.323$ kPa

The **half-life** of a substance is the time required for the amount of the substance to decrease to half of the initial amount. It is typically used when discussing radioactive decay but can be used in any exponential decay situation. As an example, Carbon-14, a radioactive element present in living materials on Earth, begins decaying at the moment of death and can be used to date once-living materials such as bone and wood. Carbon-14 decreases exponentially with a half-life of 5730 years. A quantity of carbon-14 will decay to half of its original amount after 5730 years, regardless of how big or small the original quantity was.

Homework:

1. A population of 20 fruit flies triples every month.
   A. Fill in the table showing the number of fruit flies after t months.

t	0	1	2	3	4
P(t)					

   B. Write an equation for the number of fruit flies at time t.
   C. Sketch a graph of the number of fruit flies from $0 \le t \le 6$.
   D. How many fruit flies will there be in 1.5 months?
   E. Estimate how long it will take to have 1000 fruit flies using your calculator.

2. A rancher estimates that there are 35 rabbits on his land. Every five weeks, there are 4 times the previous number of rabbits.
   A. Fill in the table showing the number of rabbits after t weeks.

t	0	5	10	15	20
P(t)					

   B. Write an equation for the number of rabbits on the rancher's land at time t.
   C. How many rabbits will there be in 3 months?
   D. Sketch a graph of the number of rabbits on the land from $0 \le t \le 12$.
   E. Estimate how long it will take to have 100 rabbits on the land using your calculator.

3. In the old science fiction horror movie <u>Island of Terror</u>, some scientists accidentally create a silicate monster that appears to be dividing (and doubling) every six hours.
   A. Fill in the table showing the number of monsters at time t (in hours)

t	0	6	12	18	24
P(t)					

   B. Plot the points and connect them in a smooth curve.
   C. Write a function for P(t) that gives the number of monsters at time t.
   D. Use the function to find the number of monsters after 8 hours.
   E. Use the function to find the number of monsters after 2 days.

4. A certain strain of bacteria that is growing on your kitchen counter doubles every 5 minutes. Assume that you start with only one bacterium.
   A. Write an equation that gives the number of bacteria on the counter after t minutes?
   B. How many bacteria could be present at the end of 86 minutes?

5. The tuition at the University of Florida was $125.91 per credit hour in 2008. Tuition has increased at a rate of approximately 14% a year since 2008.
   A. Find an equation for the tuition per credit hour where t is time in years after 2008.
   B. What is the cost per credit hour in 2013?

C. Estimate when the tuition will be $300 per credit hour.

6. Suppose that on average, one *Ixodes scapularis* tick has 1000 offspring every 25 days.
   A. Find an equation for the number of ticks after t days.
   B. How many ticks are present after 31 days?

7. In 1985, there were 285 cell phone subscribers in the small town of Chamberville. The number of subscribers increased by 75% per year after 1985.
   A. Write an equation for the number of cell phone subscribers in Chamberville t years after 1985.
   B. How many cell phone subscribers were in Chamberville in 1994?

8. In 1999, there were 5078 internet service providers. Analysts expected the number to double every five years.
   A. Write a function for the number of internet service providers t years after 1999.
   B. According to your equation, how many service providers would there be in 2012? Do you think there are that many? Why or why not?

9. During a period of rapid inflation, prices rose by 8% over 6 months. At the beginning of the inflationary period, a loaf of bread cost $2.79.
   A. Write a function that gives the price of a loaf of bread t years after inflation begins.
   B. How much would a loaf of bread cost after 15 months? After 2 years?

10. A person's current salary is $40000. The person will receive a raise of 5% every year that they work at the company.
   A. Write a function that gives the person's salary after t years of working for this company.
   B. If the person retires after 30 years, what was their salary at retirement?

11. Sam invests $7000 in an account that pays 3.9% interest compounded annually. How much will Sam have after 3 years? After 10 years?

12. Cathy invests $350 in an account that pays 2.5% interest compounded semi-annually.
   A. Write an equation that gives the amount in the account after t years.
   B. How much will be in the account after 5 years?

13. George borrows $2000 at an interest rate of 6% compounded daily. How much will he owe at the end of one year? How much would he owe at the end of 4 years?

14. Matt bought a new car at a cost of $25,000. The car depreciates (loses) approximately 15% of its value each year.
   A. Fill in the table showing the value of the car after t years.

t	0	1	2	3	4
P(t)					

   B. Write an equation for the value of the car at time t.
   C. Sketch a graph of the value of the car for $0 \le t \le 8$ years.
   D. How much will the car be worth after 3.5 years?
   E. Estimate how long it will take for the car to lose half of its value using your calculator.

15. The pesticide DDT was widely used in the United States until its ban in 1972. DDT is toxic to a wide range of animals and aquatic life, and is suspected to cause cancer in humans. The *half-life* of DDT is 15 years. If there was 150 grams of DDT present in 1972, write an equation for the amount of DDT present at any time t where t is years after 1972.

16. A fossil is found in 2010. The amount of carbon present in the fossil is 30% of the original amount. The half-life of carbon-14 is 5730 years. How old is the fossil?

17. Seventy grams of Plutonium-239, which has a **half-life** of 200 years, is leaked into soil.
    A. Write an equation for the amount of Plutonium-239 in the soil after t years.
    B. If the soil is not safe for plant-life until there is less than 15 grams of Plutonium-239, how much time will need to pass?

18. During the Vietnam War, the United States military sprayed millions of gallons of herbicides, including Agent Orange, on parts of Vietnam. After Agent Orange exposure, this dangerous chemical begins to accumulate in the fatty tissues of the body. Dioxin (a main ingredient in Agent Orange) is a very stable chemical compound, with a half-life of about 7 years, so it stays in a person's body for a long period of time. What percent of dioxin would still be present in a person's body after 40 years?

19. A patient is prescribed 500 mg of Acetaminophen. After 2 hours, 2/3 of the Acetaminophen is still present in the patient's system.
    A. Write an equation for the amount of Acetaminophen in the patient's system at any time t in hours.
    B. How much is left after 4 hours?
    C. If the patient must take another dose when the amount decreases to less than 40 mg, how often must they take the medicine?

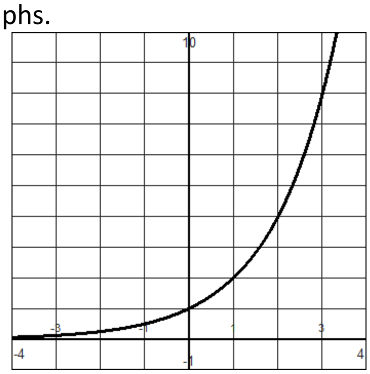
5.3 Exponential Functions

An exponential function can be written in the form $f(x) = a \cdot b^x$, where $b > 0$, $b \neq 1$, $a \neq 0$. The base of an exponential function is b. The constant a is a stretch or shrink factor.
The graphs of exponential functions have one of two basic shapes depending on whether $0 < b < 1$ or $b > 1$.

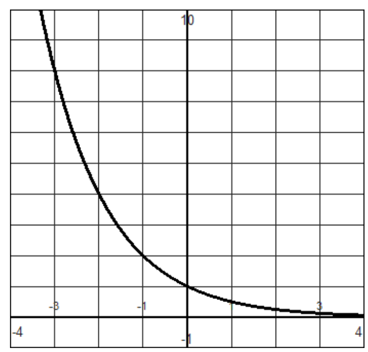
Properties of the graphs of exponential functions:

1. If $0 < b < 1$, then the graph is a decreasing function (exponential decay model). If $b > 1$, then the graph is an increasing function (exponential growth model).
2. The domain is all real numbers.
3. The range is all positive numbers.
4. The y-intercept is (0, a). This is the initial value in the exponential growth and decay models.
5. The x-axis is a horizontal asymptote of the graph.

The graphs of $f(x) = 2^x$ and $g(x) = (\frac{1}{2})^x$ are shown below. The scale on both axes are 1 unit on both graphs.



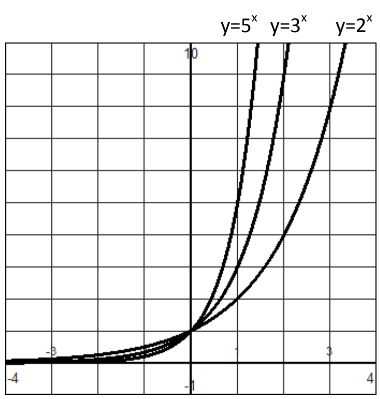
$f(x) = 2^x$



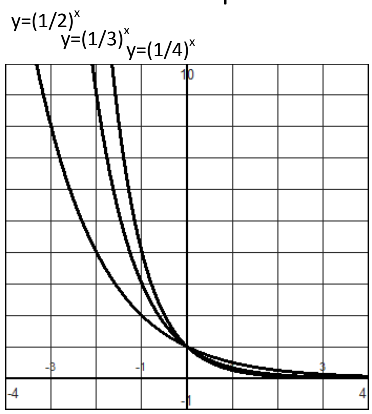
$g(x) = (\frac{1}{2})^x$

Notice that both graphs have a y-intercept of (0, 1) and a horizontal asymptote of y = 0.

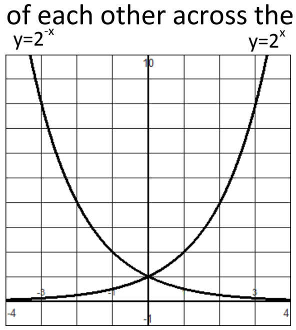
Shown on the graph below are several graphs with bases which are greater than 1. Notice that they all have the same basic shape. The larger the base, the more steeply the graph increases.



Shown on the graph below are several graphs with bases which are between 0 and 1. Notice that they all have the same basic shape. The smaller the base, the more steeply the graph decreases.



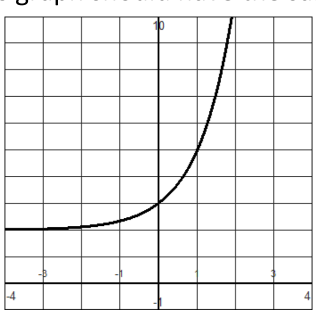
Notice that $f(x) = (\frac{1}{2})^x = 2^{-x}$. This graph and the graph of $g(x) = 2^x$ are shown below. These graphs are reflections of each other across the y-axis.



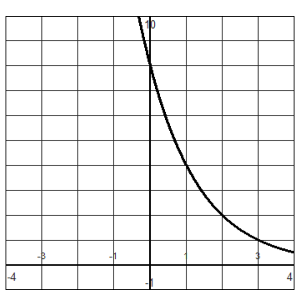
These basic shapes can be translated and transformed in the same manner as the other 8 basic shapes that were discussed earlier.

Examples:

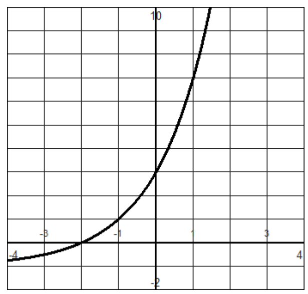
1. Sketch the graph of $f(x) = 3^x + 2$.
   This graph should have the same shape as the graph of $y = 3^x$ and be shifted 2 units upward.



2. Sketch the graph of $f(x) = (\frac{1}{2})^{x-3}$.
   This graph should have the same shape as the graph of $y = (\frac{1}{2})^x$ and be shifted 3 units to the right.



3. Sketch the graph of $f(x) = 4(2)^x - 1$.
   This graph will have the same basic shape as $y = 2^x$ but be stretched by 4 and shifted 1 unit downward.



Bases of exponential functions (base 10 and base *e*)
Bases of exponential functions can be any number larger than zero. Two bases are mostly commonly used in exponential functions. Base 10 is called the "common base" and base *e* is called the "natural base". The number *e* is an irrational number and *e* is approximately equal to 2.718. The natural base is used in many applications.

Writing an equation given a table or a graph
Given a table or a graph we would like to be able to write an exponential function which corresponds to the numerical or graphical values shown. Focusing on just equations of the form $f(x) = a \cdot b^x$, to write the equation we need to identify the values of a and b in order to write the equation.

The value of a can be found by identifying the y-intercept. The y-intercept is at the point (0, a). Recall that an exponential function has a constant multiple of b. The base b is the value that each output value is multiplied by to get the next output value.

Examples:

1. Given the table, write a formula for the exponential function described.

x	0	1	2	3
F(x)	4	12	36	108

Solution:
We can see that the y-intercept is at (0, 4). Therefore, the value of a = 4. To determine the value of b, choose two output values that have x-values 1 unit apart, divide the second output value by the first output value to get b. For instance, 12/4 = 3.

x	0	1	2	3
F(x)	4	12	36	108

x 3 x 3 x 3

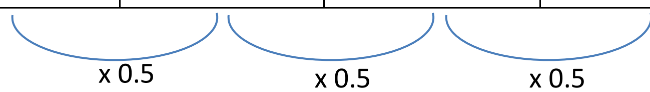
The formula for the exponential function is $F(x) = 4(3)^x$.

2. Given the table, write a formula for the exponential function described.

x	0	1	2	3
g(x)	20	10	5	2.5

Solution:
We can see that the y-intercept is at (0, 20). Therefore, the value of a = 20. To determine the value of b, choose two output values that have x-values 1 unit apart, divide the second output value by the first output value to get b. For instance, 10/20 = 0.5.

x	0	1	2	3
g(x)	20	10	5	2.5



The formula for the exponential function is $g(x) = 5(0.5)^x$ or $g(x) = 5\,(½)^x$.

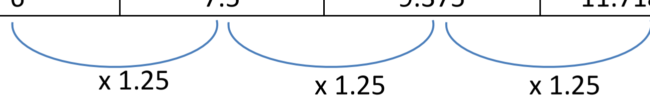
If the table does not contain consecutive values of x, then the distance between the x-values will alter the exponent of the exponential equation. Think back to the applications when the time it took for the amount to change was not 1 unit. The exponents were fractions where the denominator was the time it took for the amount to be multiplied by the constant multiple.

Example:
Given the table, write a formula for the exponential function described.

x	0	2	4	6
g(x)	6	7.5	9.375	11.71875

Solution:
We can see that the y-intercept is at (0, 6). Therefore, the value of a = 6. Notice that the table contains x-values that increase in increments of 2. To determine the value of b, choose two consecutive output values, divide the second output value by the first output value to get b. For instance, 7.5/6 = 1.25.

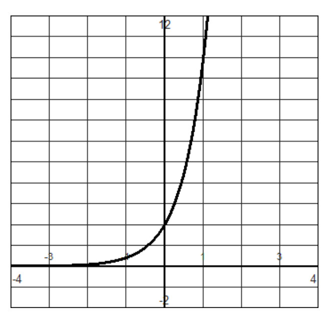
x	0	2	4	6
g(x)	6	7.5	9.375	11.71875



The formula for the exponential function is $g(x) = 6(1.25)^{x/2}$.

The same process can be used to write the equation of an exponential function of a graph. You need to identify the y-intercept and one other point on the graph.

Example:
Write the equation of the graph shown to the right.
Solution:
The graph has a y-intercept at (0, 2) and a point at (1, 10). The value of a = 2 and the value of b = 10/2 = 5. Thus, the equation of the graph is $y = 2\,(5)^x$.



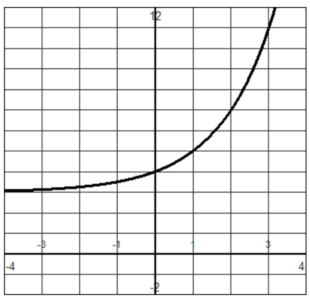
Homework:

Sketch graphs of the following functions.

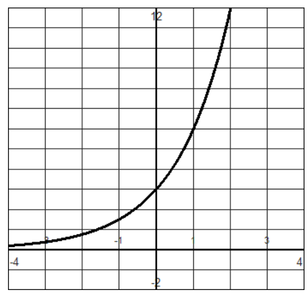
1. $f(x) = 4^x$
2. $g(x) = (1/3)^x$
3. $h(x) = 4(2)^x$
4. $F(x) = 3(1/2)^x$
5. $f(x) = ½ (3)^x$
6. $g(x) = -2^x$
7. $H(x) = 5^{x+2}$
8. $f(x) = (1/4)^x + 2$
9. $f(x) = e^x$
10. $F(x) = 10^x – 1$

11. Describe each graph shown as a transformation of $f(x) = 2^x$. Write an equation of the graphs shown.

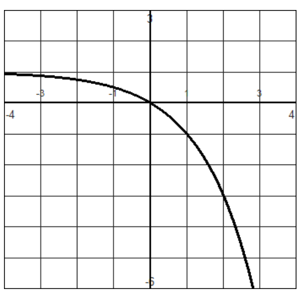
A.



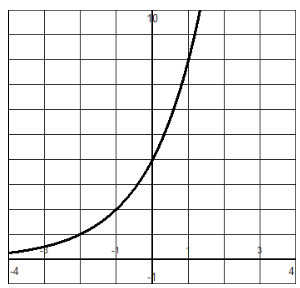
B.



C.



D.



For each of the following, identify the y-intercept and tell whether the graph is increasing or decreasing.

12. $F(x) = 6.2(0.2)^x$

13. $G(x) = 2(1.7)^x$

14. $h(x) = \frac{2}{3}(4)^x$

15. $f(x) = \frac{5}{2}\left(\frac{3}{4}\right)^x$

Write an equation for the exponential functions represented in the tables shown.

16.

x	0	1	2	3
g(x)	3	7.5	18.75	46.875

17.

x	0	1	2	3
f(x)	32	16	8	4

18.

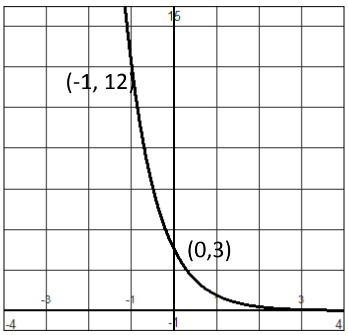
x	0	4	8	12
f(x)	40	30	22.5	16.875

19.

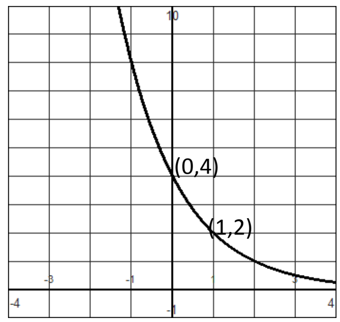
x	0	10	20	30
g(x)	100	500	2500	12500

Write an exponential function for each of the graphs shown.

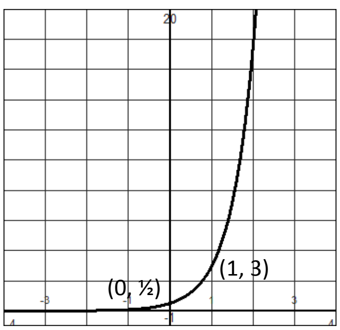
20.



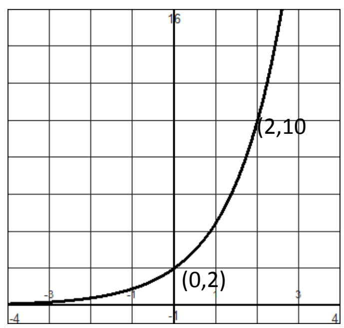
21.



22.



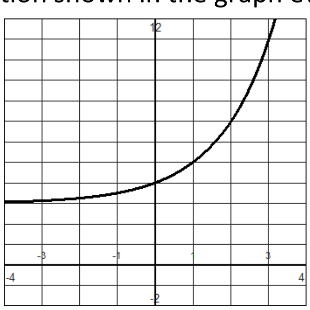
23.



24. For $f(x) = 6^x$, find:
   A. f(4) B. f(3) - f(1) C. f(a+h) D. $\frac{f(a+h)-f(a)}{h}$

25. For $f(x) = 2^{3x} + 1$, find:
   A. f(2) B. 3f(2) C. f(a+h) D. $\frac{f(a+h)-f(a)}{h}$

26. Is the function shown in the graph even, odd, or neither?



5.4 Logarithmic Functions

Logarithms were introduced by John Napier in the early 17th century as a means to simplify calculations. They were rapidly adopted by navigators, scientists, engineers, and others to perform computations more easily.

Definition: In mathematics, the **logarithm** (or **log**) of a number *x* in base *b* is the power *(y)* to which the base *b* must be raised to obtain the number *x*. The notation for a logarithm is $\log_b(x) = y$. Equivalently, $\log_b(x) = y$ if and only if $b^y = x$ where $b > 0$ and $x > 0$.

Therefore, logarithms are exponents. For example, the logarithm of 1000 to the base 10 is the number 3, because 10 raised to the power of 3 is 1000. Or, the logarithm of 81 to the base 3 is 4, because 3 raised to the power of 4 is 81.

It is important to be able to convert from exponential notation to logarithmic notation.

Examples:

1. Rewrite the following in logarithmic notation.
   A. $4^3 = 64$
      The base is 4 and the exponent is 3 so $\log_4(64) = 3$.

   B. $2^4 = 16$
      The base is 2 and the exponent is 4 so $\log_2(16) = 4$.

   C. $h^d = k$
      The base is h and the exponent is d so $\log_h(k) = d$.

2. Rewrite the following in exponential notation.
   A. $\log_5(125) = 3$
      The base is 5 and the exponent is 3 so $5^3 = 125$.

   B. $\log_9(1) = 0$
      The base is 9 and the exponent is 0 so $9^0 = 1$.

   C. $\log_m(p) = w$
      The base is m and the exponent is w so $m^w = p$.

   D. $\log_b(b) = 1$
      This is rewritten as $b^1 = b$.

The logarithm base 10 is called the **common logarithm** and can be written without writing the base number in the logarithm notation. Thus, $\log_{10}(x) = \log(x)$. If there is no base shown, it is assumed to be base 10.

The logarithm base e is called the **natural logarithm** and it written as ln(x). Thus, $\log_e(x) = \ln(x)$.

Both common and natural logarithm functions are built-in functions on the graphing calculator. We can use the definition to compute certain logarithms without a calculator. Remember that logarithms are exponents.

Examples:

Find each logarithm without using a calculator.

1. $\log_7(49)$
   The logarithm is the exponent needed to make $7^? = 49$. Since $7^2 = 49$, $\log_7(49) = 2$.
2. $\log_{25}(5)$
   The logarithm is the exponent needed to make $25^? = 5$. Since $25^{1/2} = 5$, $\log_{25}(5) = ½$.
3. $\log_3(3)$
   The logarithm is the exponent needed to make $3^? = 3$. Since $3^1 = 3$, $\log_3(3) = 1$.
4. $\log(1)$
   The logarithm is the exponent needed to make $10^? = 1$. Since $10^0 = 1$, $\log_{10}(1) = 0$.
5. $\log_4(4^5)$
   The logarithm is the exponent needed to make $4^? = 4^5$. Since $4^5 = 4^5$, $\log_4(4^5) = 5$.

Common logarithms and natural logarithms are built-in functions on the calculator. These can be approximated using the calculator buttons [log] or [ln].

Example: Approximate log(9) and ln(230) to the nearest thousandth using the calculator.
log(9) = 0.954 and ln(230) = 5.438

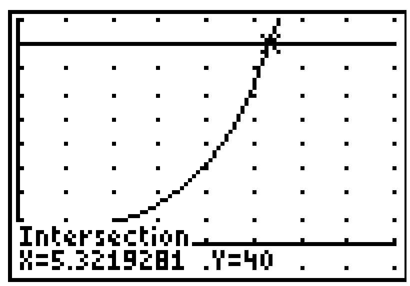
To approximate values of logarithms that are not identifiable exponents, we can use graphs or a formula or some calculators can approximate the values.

Using a graph to approximate a logarithm:

1. Rewrite the logarithm in exponential form.
2. Graph both sides of the equation.
3. Find the intersection point of the two graphs.

Example: Approximate $\log_2(40)$ to the nearest hundredth.

Solution:
$\log_2(40) = x$
Rewrite: $2^x = 40$
Graph $y_1 = 2^x$ and $y_2 = 40$.
Find the x-value of the intersection point.
x = 5.32
So, $\log_2(40) = 5.32$.



To find logarithms with bases other than 10 or *e*, the bases can be converted to 10 or *e* using the change-of-base formula.

Change-of-base formula: To change the base of a logarithm to another base, we can use the formula:

$$log_b(x) = \frac{log_a(x)}{log_a(b)}$$

Example: Approximate $\log_2(40)$ to the nearest hundredth.

Solution: Rewrite using the change-of-base formula using either base 10 or base *e*. Evaluate with your calculator.

$log_2(40) = \frac{log_{10}(40)}{log_{10}(2)} = \frac{ln(40)}{ln(2)}$ = 5.32

Some graphing calculators can find logarithms of any base.

Homework:

Rewrite each of the following in logarithmic form.

1. $4^2 = 16$
2. $5^3 = 125$
3. $7^0 = 1$
4. $2^{-1} = ½$
5. $36^{1/2} = 6$
6. $H^4 = d$
7. $x^y = z$
8. $m^7 = 3$

Rewrite each of the following in exponential form.

9. $\log_2(32) = 5$
10. $\log_5(5) = 1$
11. $\log_b (m) = 3$
12. $\log_p(30) = 2$
13. log(t) = n
14. ln(9) = v
15. $\log_6(6^{-4}) = -4$
16. $\text{Log}_{100}(10) = ½$

Use your calculator to approximate the following to the nearest hundredth.

17. log (19)
18. log(300)
19. ln(45)
20. ln(112)

Find the exact value of the following without the use of a calculator.

21. $\log_3(27)$
22. $\log_9(81)$
23. $\log_4(64)$
24. $\log_4(2)$
25. $\log_7(7^8)$
26. $\ln(e^7)$
27. $\log_{11}(1)$
28. log(10000)
29. ln(e)
30. $\log_{125}(5)$

Rewrite the following using the change-of-base formula and evaluate if possible.

31. $\log_4(22)$
32. $\log_6(121)$
33. $\log_{12}(400)$
34. $\log_3(18)$
35. $\log_b(37)$
36. $\log_f(g)$

Find the following using a graph.

37. $\log_5(70)$
38. $\log_7(100)$
39. log(78)
40. ln(78)

5.5 Solving Exponential and Logarithmic Equations

Most exponential and logarithmic equations can be solved by isolating the exponential term or logarithmic term and then rewriting the equation in the other form.

There are two methods for solving exponential equations. One method is fairly simple, but requires a very special form of the exponential equation. The other will work on more complicated exponential equations.

Let's start off by looking at the simpler method. This method will use the following fact about exponential functions. If $b^x = b^y$, then x = y. Note that this fact requires that the base in both exponentials to be the same. If it isn't then this method won't work. Let's look at a few examples.

Examples:

1. Solve $5^{3x} = 5^{21}$ for x.

   Solution:

$5^{3x} = 5^{21}$	The bases are equal. Therefore, the exponents must be equal.
$3x = 21$	Setting the exponents equal
$x = 7$	Divide both sides by 3 to solve for x

2. Solve $4^{3t-1} = 4^{2t+7}$ for t.

   Solution:

$4^{3t-1} = 4^{2t+7}$	The bases are equal. Therefore, the exponents must be equal.
$3t - 1 = 2t + 7$	Setting the exponents equal
$t - 1 = 7$	Subtracting 2t from both sides
$t = 8$	Adding 1 to both sides.

3. Solve $16^{x+1} = 8^{x-3}$ for x.

   Solution:

$16^{x+1} = 8^{x-3}$	Notice that the bases are not equal but both are powers of 2.
$(2^4)^{x+1} = (2^3)^{x-3}$	Rewrite each side so they have the same base
$2^{4x+4} = 2^{3x-9}$	Apply the property of exponents to raise a power to a power
$4x + 4 = 3x - 9$	Bases were equal so set exponents equal
$x + 4 = -9$	Subtract 3x from both sides of the equation
$x = -13$	Subtract 4 from both sides of the equation

Not all exponential equations can be solved with this method. For instance, $6^x = 19$ cannot be rewritten so that both sides of the equation have the same base. Therefore, we need another method of solution.

Steps to solve an exponential equation algebraically:

1. Isolate the power on one side of the equation.
2. Rewrite the equation in logarithmic form.
3. Evaluate the logarithm, using a calculator if necessary.
4. Solve for the variable if needed.

Examples:

1. Solve $3(2^x) + 4 = 19$ for x.

   Solution:
   $3(2^x) + 4 = 19$
   $3(2^x) = 15$ Subtract 4 from both sides
   $(2^x) = 5$ Divide both sides by 3
   $\log_2(5) = x$ Rewrite in logarithmic form
   $x = 2.322$ Evaluate the logarithm.

2. Solve $2(5^{2x-1}) = 24$ for x.

   Solution:
   $2(5^{2x-1}) = 24$
   $5^{2x-1} = 12$ Divide both sides by 2
   $\log_5(12)= 2x – 1$ Rewrite in logarithmic form
   $1.544 = 2x – 1$ Evaluate the logarithm
   $2.544 = 2x$ Add 1 to both sides
   $x = 1.272$ Divide both sides by 2

3. Solve $e^{3x + 5} = 40$ for x.

   Solution:
   $e^{3x + 5} = 40$ Note that the term with the power is isolated
   $\ln(40) = 3x + 5$ Rewrite in logarithmic form
   $3.689 = 3x + 5$ Evaluate the logarithm
   $-1.311 = 3x$ Subtract 5 from both sides
   $x = -0.437$ Divide both sides by 3

Logarithmic equations are solved in a similar manner to exponential equations.

Steps to solve a logarithmic equation algebraically:

1. Isolate the logarithm on one side of the equation.
2. Rewrite the equation in exponential form.
3. Evaluate the power, using a calculator if necessary.
4. Solve for the variable if needed.

Examples:

1. Solve $2\log_4(x) + 3 = 9$ for x.

   Solution:
   $2\log_4(x) + 3 = 9$
   $2\log_4(x) = 6$ Subtract 3 from both sides
   $\log_4(x) = 3$ Divide both sides by 2
   $4^3 = x$ Rewrite in exponential form
   $x = 64$

2. Solve $-3\log_2(x) – 6 = 9$ for x.

Solution:

$-3\log_2(x) - 6 = 9$	
$-3\log_2(x) = 15$	Add 6 to both sides
$\log_2(x) = -5$	Divide both sides by -3
$2^{-5} = x$	Rewrite in exponential form
$x = \frac{1}{32}$	

3. Solve ln(2x + 5) = 4 for x.

Solution:

$\ln(2x + 5) = 4$	Note the logarithm is isolated
$e^4 = 2x + 5$	Rewrite in exponential form
$54.598 = 2x + 5$	Evaluate the exponent
$49.598 = 2x$	Subtract 5 from both sides
$x = 24.799$	Divide both sides by 2

All exponential and logarithmic equations can be solved graphically. These equations are not usually solved numerically because the solutions are rarely integer values.

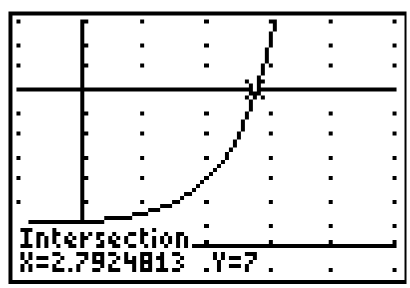
Some examples of solving graphically are shown below. Solving these equations graphically require the same steps as solving any equation graphically.

Steps to solve graphically:

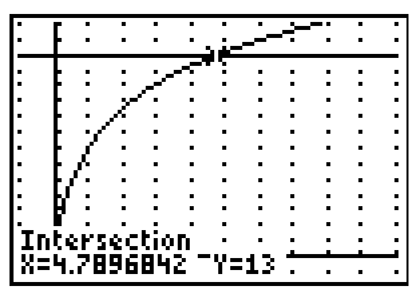
1. Graph y_1 = left side of the equation and y_2 = right side of the equation.
2. Set an appropriate viewing window to see the intersection point.
3. Find the intersection point using the intersection feature of the calculator.
4. The solution will be the x-value of the intersection point.

Examples:

1. Solve $2(4^{x-2}) + 1 = 7$ graphically.
   Solution:
   Graph $y_1 = 2(4^{x-2}) + 1$ and $y_2 = 7$. Set a viewing window where the intersection point is visible. Use the intersection feature to find the intersection of the two graphs. Solution is x = 2.792.



2. Solve $3\log_2(4x + 1) = 13$
   Solution:
   Graph $y_1 = 3\log_2(4x + 1)$ (you may need to use the change-of-base formula to graph this as $y_1 = 3\,\frac{\log(4x+1)}{\log(2)}$) and $y_2 = 13$. Set a viewing window where the intersection point is visible. Use the intersection feature to find the intersection of the two graphs. Solution is x = 4.79.



Homework:

Solve the following equations algebraically.

1. $2^{3x-1} = 2^{12}$
2. $4^{x+2} = 4^{9}$
3. $3^{2x-5} = 27$
4. $10^{-2x+5} = 1000$
5. $4^{x-1} = 16^{2x-9}$
6. $3^{4x} = 9^{x+4}$
7. $25^{x+6} = 125^{2x}$
8. $9^{2x+1} = 27^{x-1}$

9. $7^{x} = 12$
10. $3^{x} = 120$
11. $3(2)^{x+1} = 21$
12. $4(10)^{2x+1} = 240$
13. $10^{-0.2x} + 2 = 20$
14. $e^{-2x} = 45$
15. $5(3)^{5x-6} = 33$
16. $2(6)^{1.2x+5} - 7 = 11$
17. $7(½)^{3x} - 5 = 16$
18. $300(0.75)^{x/4} = 100$
19. $50(1.04)^{12x} = 80$
20. $100e^{0.03t} = 250$

21. A certain strain of bacteria that is growing on your kitchen counter doubles every 5 minutes. Assume that you start with only one bacterium. How long will it take to have 1000 bacteria?

22. The tuition at the University of Florida was \$125.91 per credit hour in 2008. Tuition has increased at a rate of approximately 14% a year since 2008. If tuition continues to increase at this rate, in what year will tuition be \$350 per credit hour?

23. During a period of rapid inflation, prices rose by 7% over 6 months. At the beginning of the inflationary period, a gallon of milk cost \$3.59. How long will it take for the price of a gallon of milk to reach \$5.00?

24. A person's current salary is \$45000. The person will receive a raise of 5% every year that they work at the company. How long will the person have to work at the company to have a salary of \$70,000?

25. Sam invests \$7000 in an account that pays 3.9% interest compounded annually. How long will it take for Sam's money to double? To triple?

26. Harry buys a car for \$56000. The value of his car each year is 5/6ths of its value the previous year. How long will it take for his car to be worth half of its original value? How long will it take to be worth \$7000?

27. The growth of a colony of bacteria is given by the equation, $P(t) = 500e^{0.195t}$, where t is in hours. How long will it take before there are 7500 bacteria in the colony?

28. Sodium-24 is a radioactive isotope that is used in diagnosing circulatory diseases. It decays at a rate of 4.73% per hour. Technicians inject a quantity of sodium-24 into a patient's bloodstream. How long will it take for 75% of the isotope to decay?

5.6 Logarithmic Functions, Graphs, and Properties of Logarithms

Recall that the basic exponential function had the form $y = b^x$. Its inverse would then be $x = b^y$ but this by definition is $y = \log_b(x)$. Therefore, exponential and logarithmic functions are inverses of each other.

Inverses undo each other so $\log_b(b^x) = x$ and $b^{log_b(x)} = x$. The bases must be the same on the exponential and logarithmic functions for them to be inverses.

Example:

Simplify:

A. $\log_3(3^w)$
   Since both the exponential and logarithm are base 3, they are inverses, so $\log_3(3^w) = w$.
B. $\log(10^6)$
   Since both the exponential and logarithm are base 10, $\log(10^6) = 6$.
C. $7^{log_7(11)}$
   Since both bases are 7, this simplifies to 11.
   $7^{log_7(11)} = 11$
D. $e^{ln(m)}$
   Since ln has a base of e, these are inverses and $e^{ln(m)} = m$.

Graphs of Logarithmic Functions

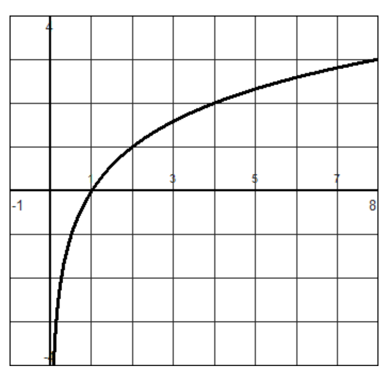
To graph a logarithmic function, we can rewrite the equation as an exponential equation and then plot points. Another method is to use the fact the graphs of inverse functions are reflected about the line y = x. Thus, we could graph the exponential function which is the inverse of the logarithmic function and reflect it across the line y = x or equivalently, we could find points on the exponential function and interchange the domain and range to get points on the logarithmic function.

Using the fact the logarithms and exponential functions are inverses, graphs of logarithmic functions will have the following properties.

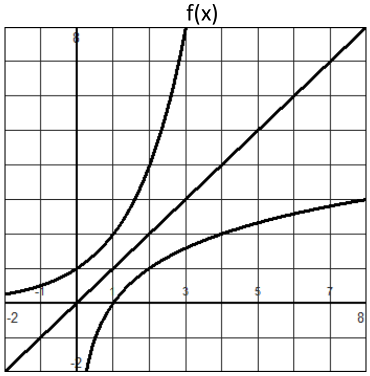
1. Domain: all positive real numbers
2. Range: all real numbers
3. x-intercept at (1, 0)
4. Vertical asymptote at x = 0

The graph of $f(x) = \log_2(x)$ can be graphed by writing the equation in the equivalent form $2^y = x$ and making a table of values.

x	y
½	-1
1	0
2	1
4	2
8	3



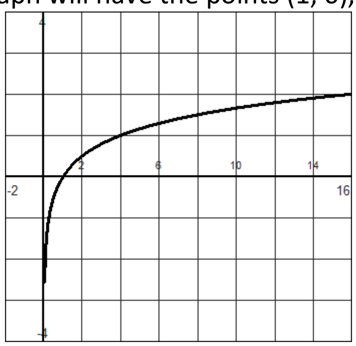
The graphs of $f(x) = 2^x$ and $g(x) = \log_2(x)$ are shown on the graph below. Notice that they are reflections of each other across the line y = x.



Example:

Graph $f(x) = \log_4(x)$.

The graph will have the points (1, 0), (4, 1), (16, 2) and (1/4, -1).



There are three properties of logarithms that allow us to rewrite several logarithms as a single logarithm or allow us to split a single logarithm into multiple simpler logarithms. The properties follow from the laws of exponents.

Properties of Logarithms

If x > 0, y > 0, and b > 0, then

1. $\log_b(xy) = \log_b(x) + \log_b(y)$

2. $\log_b(x/y) = \log_b(x) – \log_b(y)$

3. $\log_b(x^r) = r \log_b(x)$

The properties can be used to write one logarithm in terms of simpler logarithms.

Examples:

1. Rewrite $\log_b(xy^4)$ in terms of simpler logs.

Solution:

$\text{Log}_b(x) + \log_b(y^4)$	Using property 1
$\log_b(x) + 4\log_b(y)$	Using property 3

2. Rewrite $\log_2\left(\frac{\sqrt{x}}{3y}\right)$ in terms of simpler logs.

Solution:

$\log_2\sqrt{x} - \log_2(3y)$	Using property 2
$\frac{1}{2}\log_2(x) - \log_2(3) - \log_2(y)$	Using properties 1 and 3

The properties can also be used to combine logarithms with the same base into a single logarithm as seen in the following examples.

Examples:

1. Express $2(\log_b(x) - \log_b(w))$ as a single log.

Solution:

$2\log_b(x/w)$	Using property 2
$\text{Log}_b(x/w)^2$	Using property 3

2. Express $3\log_5(d) + 2\log_5(h+1)$ as a single log.

Solution:

$\log_5(d^3) + \log_5(h+1)^2$	Using property 3
$\log_5(d^3(h+1)^2)$	Using property 1

These properties can be used to combine logarithms in an equation to get a single log so that we can solve the equation as shown previously. If there are multiple logarithms with the same base in the equation, combine the logs into a single log then solve the equation as before.

Example:

Solve $\log_4(x+2) – \log_4(x) = 3$ for x.

Solution:

$\log_4(x+2) – \log_4(x) = 3$	Using property 2 combine the logs
$\log_4\left(\frac{x+2}{x}\right) = 3$	Rewrite in exponential form
$4^3 = \frac{x+2}{x}$	Simplify
$64 = \frac{x+2}{x}$	Multiply both sides by x or cross multiply
$64x = x + 2$	Subtract x from both sides
$63x = 2$	Divide by 63
$x = 2/63$	

Homework:

Simplify the following:

1. $\log_5(5^7)$

2. $\log(10^9)$

3. $\log_8(8^{-2})$

4. $\ln(e^{11})$

5. $\log_2(2^t)$

6. $\log_{20}(20^{1/3})$

7. $4^{log_4(12)}$

8. $7^{log_7(2)}$

9. $e^{ln(6)}$

10. $e^{ln(x)}$

11. $10^{log(13)}$

12. $m^{log_m(n)}$

13. Sketch a graph of $f(x) = \log_3(x)$.

14. Sketch a graph of $f(x) = \log(x)$.

15. Sketch a graph of $f(x) = \ln(x)$.

16. Sketch a graph of $f(x) = \log_{1/2}(x)$.

Use properties of logarithms to write the following in terms of simpler logarithms.

17. $\log_3(7x^6)$

18. $\log_a(x^2\ y^3)$

19. $\ln(mn)^2$

20. $\log_b[(x+3)^4\ (y+6)^5]$

21. $\log_6\left(\frac{\sqrt{3x}}{y}\right)$

22. $\log_2\left(\frac{4t}{r}\right)$

23. $\ln\left(\sqrt[3]{\frac{x}{y}}\right)$

24. $\log\left(\frac{g}{h^5}\right)$

Use properties of logarithms to write the following as a single logarithm.

25. $\log_b(z) - \log_b(w)$

26. $2\log_b(x) - \log_b(y)$

27. $2\log_4(x) + 3\log_4(w)$

28. $\frac{1}{2}(\log_3(x) + \log_3(p))$

29. $\log_b(x) - \log_b(w) + \log_b(d)$

30. $3\log_b(x) + \log_b(y) - \frac{1}{4}\log_b(z)$

Solve the following equations.

31. $\log_2(x+2) - \log_2(x) = 4$

32. Solve $\log(2x+1) - \log(x-5) = 2$

33. $\log_2(x + 3) + \log_2(x) = 2$

34. $\log_4(x+2) + \log_4(7) = 3$

35. Based on data, the expected life span of people in the United States can be described by the function $f(x) = 12.734 \ln(x) + 17.875$, where x is the number of years from 1900 to the person's birth year.
   A. Find the expected life span of a person born in 1965.
   B. Determine the birth year for which the expected life span is 67 years.
   C. What is the expected life span of a person born in 1991?

36. The intensity of sound is measured in Decibels, where $D = 10log\left(\frac{I}{I_0}\right)$ and I is the intensity of sound and $I_0 = 1 \times 10^{-12}$.
   A. Find the value of D in decibels of a whisper with an intensity of $115I_0$.
   B. Find the value of D in decibels of a rock concert with an intensity of $895{,}000{,}000I_0$.

37. $H(A) = 29 + 48.8 \log(A + 1)$ gives the percent of the adult height a child of age A has achieved.
   A. If a child is 10 years old, what percent of his adult height will he have reached?
   B. If the same child is 59 inches tall, how tall will he be as an adult?

38. Verify that $f^{-1}(x) = \log_3(x)$ is the inverse of $f(x) = 3^x$.

5.7 Modeling with Exponential Equations

In this section we focus on applications involving exponential functions and how to fit an exponential function to data. Exponential models often are useful in solving problems involving change in populations, pollution, temperature, bank savings, drugs in the bloodstream, and radioactive materials. We have previously looked at exponential growth and decay examples and compound interest problems. We first revisit the compound interest formulas.

Compound Interest

Compound interest is calculated each period on the **original principal and all interest accumulated during past periods**. Although the interest may be stated as a yearly rate, the compounding periods can be yearly, a specific number of times per year (monthly, quarterly, daily), or continuously.

The amount in an account if the interest is **compounded annually** can be found using the formula $A(t) = P(1+r)^t$ where A is the amount in the account, P is the principal, r is the interest rate, and t is time in years.

If interest is earned a specific number of times a year, n, the amount in the account is given by the formula $A(t) = P(1+r/n)^{nt}$. For instance, if interest is compounded quarterly, then n = 4, and if interest is compounded monthly, then n = 12. P represents the principal, r is the annual interest rate, and t is time in years.

If interest is **compounded continuously**, then the formula for the amount in an account is $A(t) = Pe^{rt}$, where P is the principal, r is the interest rate, and t is time in years.

Examples:

1. You invest $10000 at 5% interest. How much money will you have at the end of 20 years if the interest is compounded:

   A. annually?

   Solution:

   The principal is $10000 and the interest rate is 5% or 0.05.
   $A(t) = 10000(1+0.05)^t$
   $A(t) = 10000(1.05)^t$

   After 20 years, $A(20) = 10000(1.05)^{20} = \$26532.98$.

   B. monthly?

   Solution:

   The principal is $10000 and the interest rate is 5% or 0.05 and the interest is earned 12 times a year (n=12).
   $A(t) = 10000(1+0.05/12)^{t(12)}$
   $A(t) = 10000(1.004166667)^{12t}$

   After 20 years, $A(20) = 10000(1.004166667)^{12(20)} = \$27126.40$.

C. continuously?

Solution:
The principal is \$10000 and the interest rate is 5% or 0.05.
$A(t) = 10000e^{0.05t}$

After 20 years, $A(20) = 10000e^{0.05(20)}$ = \$27182.82.

2. How much would you need to invest now, to get \$15,000 in 10 years at 3% interest rate compounded quarterly?

Solution:
The interest rate is 3% or 0.03, n=4 (quarterly), and t = 10 years. We are trying to find the principal P when A = 15000.
$A(t) = P(1+.03/4)^{4(10)}$
$15000 = P(1.0075)^{40}$
$15000 = P\ (1.34835)$
P = \$11124.72

3. You have \$1,000, and want it to double in 7 years, what interest rate do you need if your money is compounded annually? What interest rate would you need if your interest is compounded continuously?

Solution:
P = 1000, A = 2000, t = 7 years, and we are trying to find the value of r.
Annually:
$A = P(1 + r)^t$
$2000 = 1000(1 + r)^7$ Divide both sides by 1000
$2 = (1 + r)^7$ Raise each side to the 1/7 power
$2^{1/7} = 1 + r$
$1.104 = 1 + r$ Subtract 1 from both sides
$0.104 = r$
Interest rate must be 10.4%.
Continuously:
$A = Pe^{rt}$
$2000 = 1000e^{r(7)}$ Divide both sides by 1000
$2 = e^{7r}$ Rewrite in logarithmic form
$\ln(2) = 7r$ Divide by 7
$r = \ln(2)/7 = 0.099$
Interest rate must be 9.9%.

Other exponential applications include growth and decay problems. We have looked at writing the equations of graphs and numerical data when the y-intercept is known. Now, let's look at several examples where the y-intercept must be found as well as the base of the exponential equation. When the y-intercept is unknown, we use a system of equations and the substitution method to solve for the the y-intercept and the base.

Examples:

1. During the 1980s the population of a certain city rose to 205,000 in 1989. The population of the city was 127,000 in 1983. Find an equation for the population of the city at any time t, where t is the number of years after 1980.

   Solution:
   We do not know the population at t = 0 (1980). We know that the population in 1983 (t = 3) was 127000 and the population in 1989 (t = 9) is 205000.
   The general equation is $P(t) = a(b)^t$.
   The point (3, 127000) gives $127000 = a(b)^3$.
   The point (9, 205000) gives $205000 = a(b)^9$.
   This is a system of equations which can be solved using substitution. Solving the first equation for a: $a = \frac{127000}{b^3}$
   Substituting into the second equation for a:
   $205000 = \frac{127000}{b^3} \cdot b^9$ Simplifying
   $205000 = 127000b^6$ Dividing both sides by 127000
   $1.61417 = b^6$ Raising both sides to the 1/6 power
   $b = 1.083$
   Solving for a: $a = \frac{127000}{b^3} = \frac{127000}{(1.083)^3} = 99981$
   The equation for the population of the city is $P(t) = 99981(1.083)^t$.

2. If left to sit on the counter, a cup of hot coffee will cool to room temperature. At 5 minutes after the coffee is poured, the coffee is 36°C above room temperature. At 14 minutes after the coffee is poured, the coffee is 24°C above room temperature. Write an equation for the amount the temperature of the coffee exceeds the temperature of the room at any time t where t is the number of minutes after the coffee is poured. If the coffee sits for 30 minutes, how much hotter is it than room temperature?

   Solution:
   We have the two points (5, 36) and (14, 24). The general equation is $T(t) = ab^t$.
   Substituting the points into the equation.
   $36 = a(b)^5$
   $24 = a(b)^{14}$
   Solving the first equation for a, we find $a = \frac{36}{b^5}$ and substituting into the second equation gives:
   $24 = \frac{36}{b^5} \cdot b^{14}$ Simplifying
   $24 = 36b^9$ Dividing by 36 on both sides
   $2/3 = b^9$ Raising both sides to the 1/9 power
   $b = (2/3)^{1/9} = 0.956$
   Solving for a: $a = \frac{36}{b^5} = \frac{36}{0.956^5} = 45.08$
   The equation that gives the difference between the coffee temperature and the room temperature is $T(t) = 45.08(0.956)^t$.

   After 30 minutes, the coffee is $T(30) = 45.08(0.956)^{30} = 11.69$°C over room temperature.

Homework:

1. Assume you have a bank account whose principal is $1000, and your bank compounds the interest twice a year at an interest rate of 5%.
   A. How much money do you have in your account at the year's end?
   B. How much will you have after 4 years?
   C. How long will it take to triple your money?

2. Assume you start a bank account with $5,000 and your bank offers an interest rate of 2.5%.
   A. How much money do you have after 2 years if the interest is compounded semi-annually?
   B. How much money do you have after 2 years if the interest is compounded continuously?

3. How much would you need to invest now, to get $75,000 in 25 years at 3% interest rate compounded quarterly?

4. How much would you need to invest now, to get $4,000 in 5 years at 4.5% interest rate compounded daily?

5. You have $400, and want it to double in 8 years, what interest rate do you need if your money is compounded semi-annually? What interest rate would you need if your interest is compounded continuously?

6. You have $2500, and want it to have $4000 in 6 years, what interest rate do you need if your money is compounded annually? What interest rate would you need if your interest is compounded continuously?

7. During the 1990s the population of a certain city rose to 100,000 in 1999. The population of the city was 78,000 in 1994. Find an equation for the population of the city at any time t, where t is the number of years after 1990.

8. A magazine started publication is 1975. The cost of the magazine was $1.25 in 1980 and $3.50 in 2000. Assuming the cost increases exponentially, find an exponential equation for the cost of the magazine at time t, where t is the number of years after the start of publication.

9. The first credit card that you got charges 12.49% interest to its customers and compounds that interest monthly. Within one day of getting your first credit card, you max out the credit limit by spending $1,200.00 . If you do not buy anything else on the card and you do not make any payments, how much money would you owe the company after 6 months?

10. If a person takes *A* milligrams of a drug at time 0, write an equation for the concentration of the drug in the bloodstream if after 1 hour there is 87.5 mg of the drug in the person's bloodstream and after 3 hours there is 42.875 mg of the drug in the person's bloodstream?

11. A species of turtle is introduced into a new habitat. After 3 years, there are 240 turtles and after 7 years, there are 450 turtles.
    A. Write an equation for the number of turtles in this habitat at any time t.
    B. How many turtles were originally introduced into the new habitat?
    C. How many turtles will there be after 10 years?

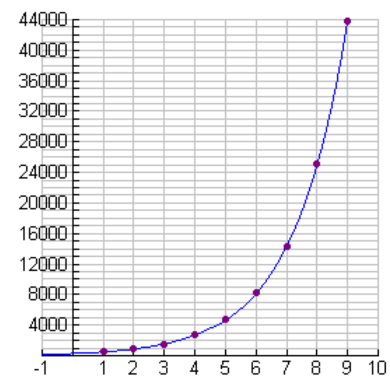
12. Pegi buys a 3-year-old car for $22,800. When the car is 6 years old it is worth $15,550.
    A. Write an equation for the value of the car (t = 0 is when the car is brand new).
    B. What was the original selling price of the car?
    C. How much will the car be worth after 8 years?

13. The data at the right shows the cooling temperatures of a freshly brewed cup of coffee after it is poured from the brewing pot into a serving cup.

(mins)	Temp (° F)
5	168.7
15	141.7
25	123.5
30	116.3

    A. Using the points at 5 minutes and 25 minutes, write an exponential equation for the temperature of the cup of coffee.
    B. Graph this equation on your graphing calculator. Discuss how well the equation fits the other points.
    C. Based upon this equation, what was the initial temperature of the coffee?
    D. When is the coffee at a temperature of 106 degrees?
    E. In 1992, a woman sued McDonald's for serving coffee at a temperature of 180° that caused her to be severely burned when the coffee spilled. An expert witness at the trial testified that liquids at 180° will cause a full thickness burn to human skin in two to seven seconds. It was stated that had the coffee been served at 155°, the liquid would have cooled and avoided the serious burns. The woman was awarded over 2.7 million dollars. As a result of this famous case, many restaurants now serve coffee at a temperature around 155°. How long should restaurants wait (after pouring the coffee from the pot) before serving coffee, to ensure that the coffee is not hotter than 155°?

14. Given the graph shown, identify two points and use those points to find the equation of the graph. The horizontal axis is years after 1980 and the vertical axis is the number of cell phone users.



15. The graph shown gives the number of bacteria after t days. Estimate two points from the graph and use those points to find the equation of the graph.